The Information Bottleneck for Optimal Prediction of the Voter Model

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mathe MACS
Mathematics for Multilevel Anticipatory Complex Systems

MAX-PLANCK-GESELLSCHAFT

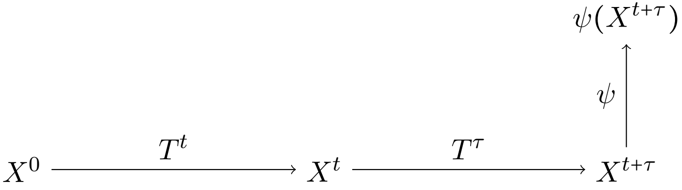
General Setting

$$X^0 \xrightarrow{\quad T^t \quad} X^t \xrightarrow{\quad T^\tau \quad} X^{t+\tau}$$

- Markovian Kernel $T(X^{t+1}|X^t)$

- Initial State $X^0 \in \Sigma$
- Current State $X^t \in \Sigma$ with Current Time $t \in \mathbb{N}$
- Future State $X^{t+\tau} \in \Sigma$ with Prediction Horizon $\tau \in \mathbb{N}$

General Setting



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- Post-measurement $\psi : \Sigma \to \mathcal{S}_\psi$ defined by $\Pr(\psi(X)|X)$

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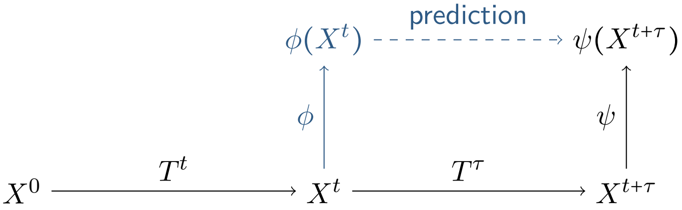

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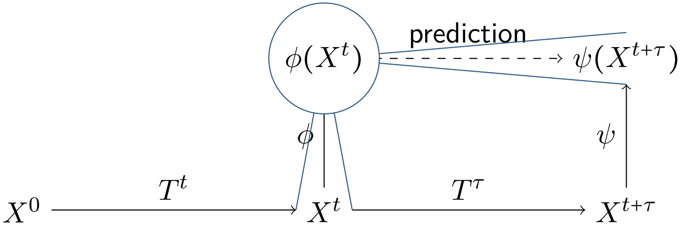
- Pre-measurement $\phi : \Sigma \to \mathcal{S}_\phi$ defined by $\Pr(\phi(X)|X)$

The Optimal Prediction Problem



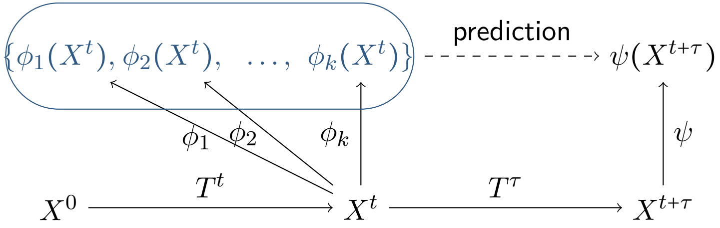
The Information Bottleneck Method [Tishby *et al.*, 1999]:

- **Maximize** Predictive Capacity $max_\phi \; I(\phi(X^t); \psi(X^{t+\tau}))$
- **Minimize** Measurement Complexity $min_\phi \; I(X^t; \phi(X^t))$

- **Minimize** the IB-variational

$$min_{\Pr(\hat{X}|X)} \quad I(X^t; \phi(X^t)) - \beta \; I(\phi(X^t); \psi(X^{t+\tau})) \quad \text{with } \beta \in \mathbb{R}^+$$

Constraining the Set of Feasible Measurements

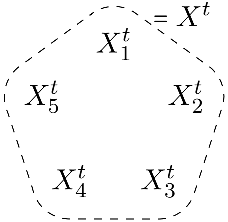


- Given a collection $\Phi = \{\phi_1, \ldots, \phi_k\}$ of *feasible* pre-measurements
- **Minimize** the IB-variational

$$min_{\phi \in \Phi} \quad I(X^t; \phi(X^t)) \ - \ \beta \ I(\phi(X^t); \psi(X^{t+\tau})) \quad \text{with } \beta \in \mathbb{R}^+$$

Application to Agent-based Systems

- Agent Set $\Omega = \{1, \ldots, N\}$
- Agent States $X_1^t \in S, \quad X_2^t \in S, \quad \ldots, \quad X_k^t \in S$
- System State $X^t = (X_1^t, X_2^t, \ldots, X_N^t) \in \Sigma = S^N$



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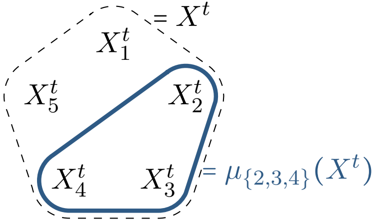
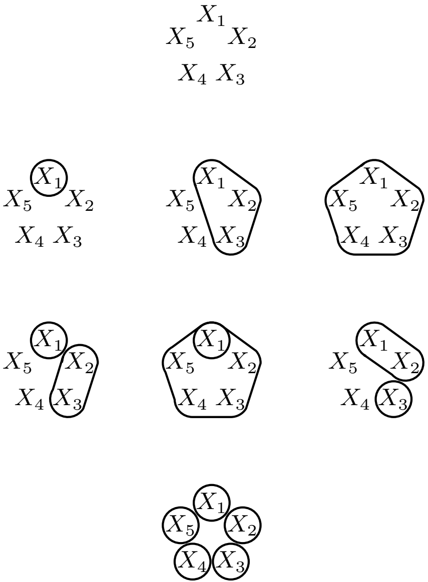
- Generic Measurement μ
  a familly of measurements $(\mu_A : \Sigma \to \mathcal{S}_\mu)$ for any $A \subset \Omega$
  such that $\Pr(\mu_A(X)|X) = \Pr(\mu_A(X)|(X_i)_{i\in A})$



The Poset of Feasible Measurements



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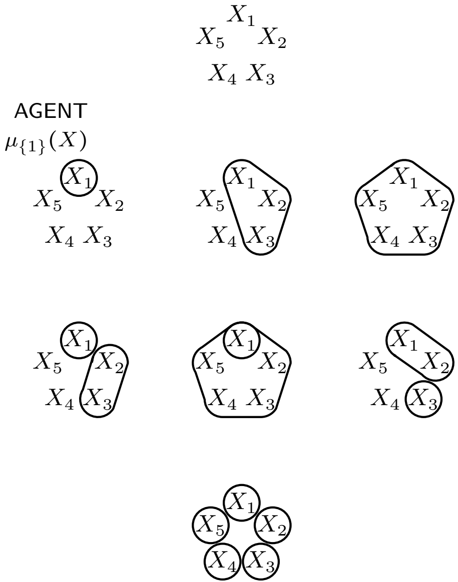

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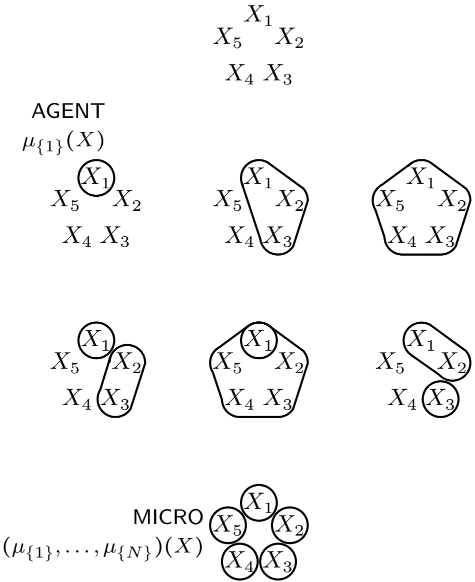

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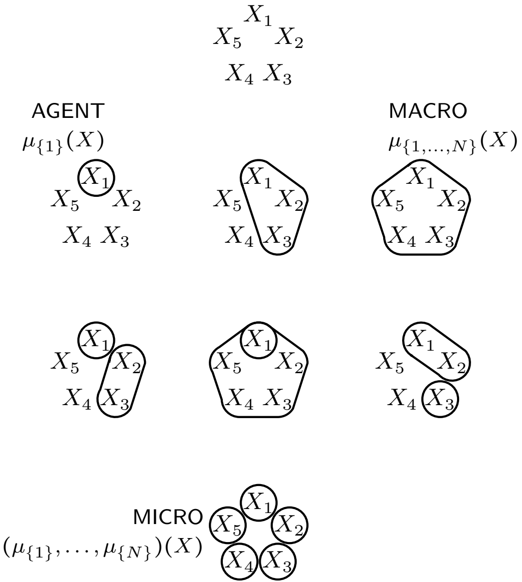

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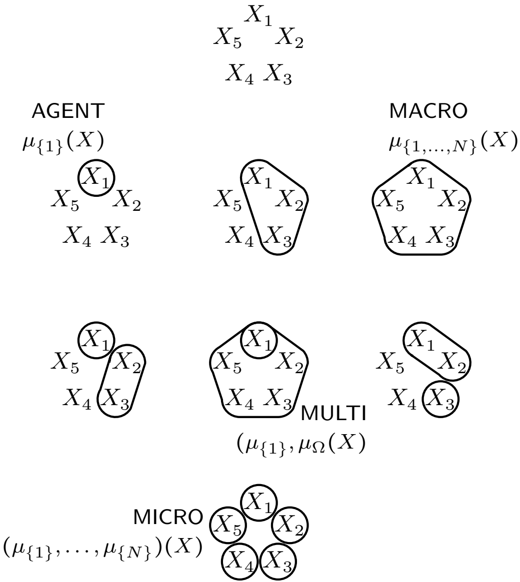

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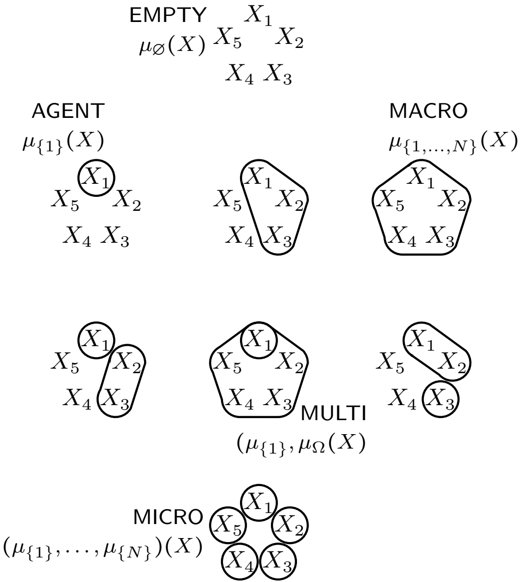

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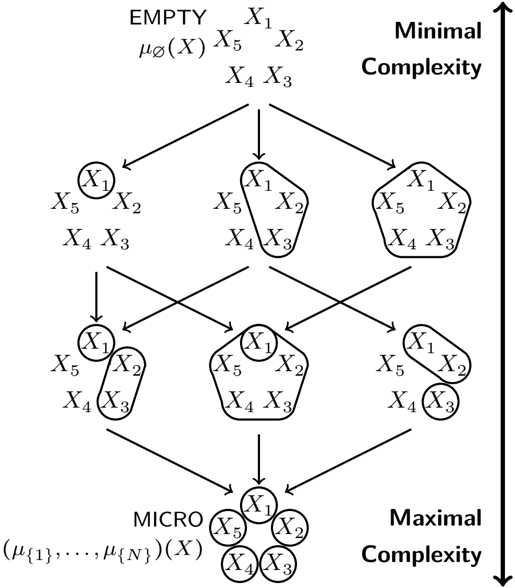


Definition 1 (Additivity)

μ additive iff $\forall A \cap B = \varnothing$,

$$H(\mu_{A\cup B}(X) \mid \mu_A(X), \mu_B(X)) = 0$$
$$H(\mu_A(X) \mid \mu_{A\cup B}(X), \mu_B(X)) = 0$$

Definition 2 (Refinement)

$\phi_1 \prec \phi_2$

iff $X \to \phi_1(X) \to \phi_2(X)$ is Markovian

iff $I(X; \phi_2(X) | \phi_1(X)) = 0$

Theorem 1 (Monotonicity)

$\phi_1 \prec \phi_2 \;\Rightarrow$

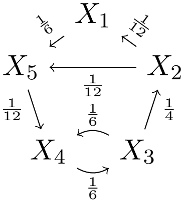
$I(\phi_1(X^t); \psi(X^{t+\tau})) \geq I(\phi_2(X^t); \psi(X^{t+\tau}))$

and $I(X^t; \phi_1(X^t)) \geq I(X^t; \phi_2(X^t))$

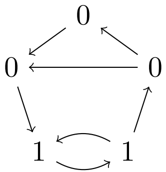
The Voter Model

- Set of Agents $\Omega = \{1, \ldots, N\}$
- State of ith Agent $X_i^t \in \{0, 1\}$
- System State $X^t = (X_1^t, \ldots, X_N^t) \in \{0, 1\}^N$
- Transitions Kernel $T(X^{t+1} | X^t)$ given by an interaction graph:

arc (i, j) selected $\Rightarrow$ $X_j^{t+1} = X_i^t$ and $\forall k \neq j,$ $X_k^{t+1} = X_k^t$



$t = 0$



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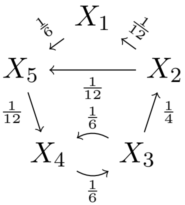
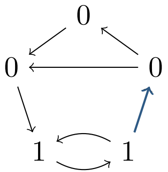


$t = 0$ $\rightarrow$ arc $(3, 2)$



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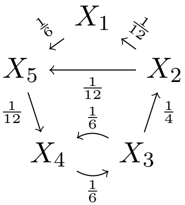
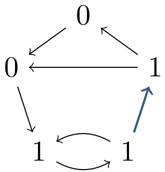


$t = 1$



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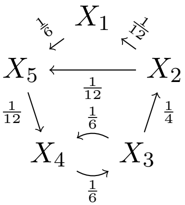
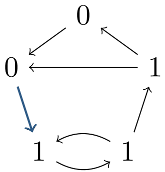


$t = 1$ $\rightarrow$ arc $(5, 4)$



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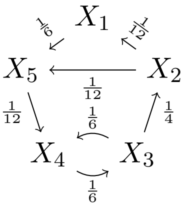

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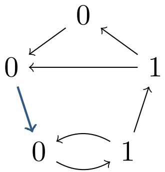

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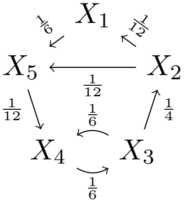
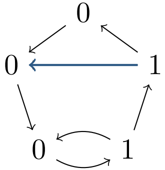


$t = 2$ $\rightarrow$ arc $(2, 5)$



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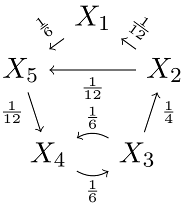

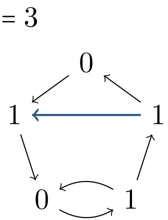

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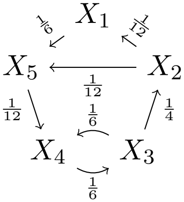
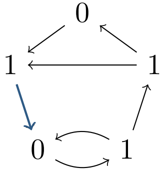


$t = 3$ $\rightarrow$ arc $(5, 4)$



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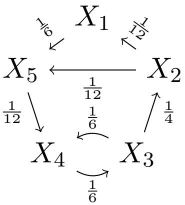

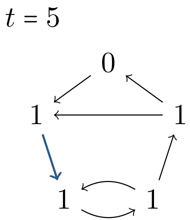

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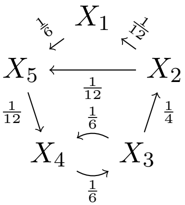
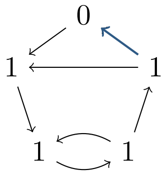


$t = 5$ $\rightarrow$ arc $(2, 1)$



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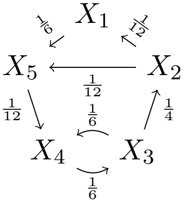

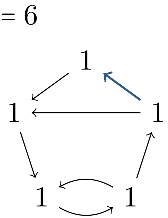

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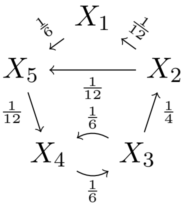
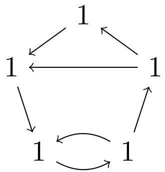


$t = 6$



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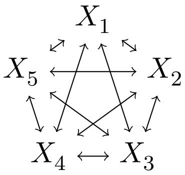
Aggregated-states in the Complete Graph

- All arcs are equally likely $\forall (i,j) \in \Omega^2, \quad \Pr(\text{arc } (i,j)) = \frac{1}{N\,(N-1)}$
- Uniform Initial State $\forall x \in \{0,1\}^N, \quad p(X^0 = x) = 2^{-N}$



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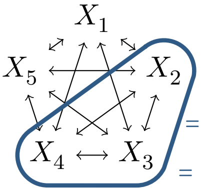
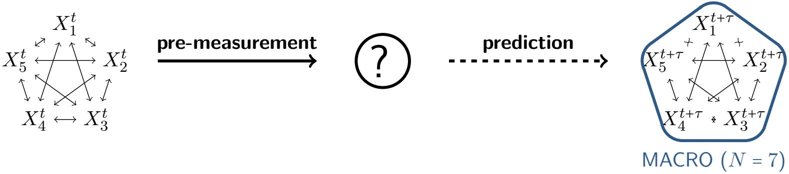
- Generic Measurement $\forall A \subset \Omega, \quad \eta_A(x) = \sum_{i \in A} x_i$
  (Aggregated-state Measurement)

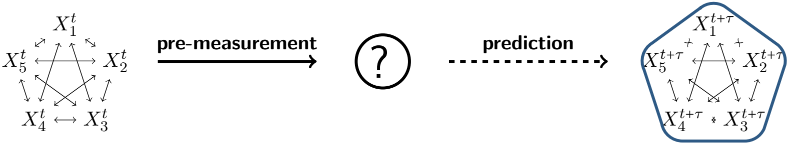


$= \mu_{\{2,3,4\}}(X)$
$= X_2 + X_3 + X_4$

Predicting the Macroscopic Measurement

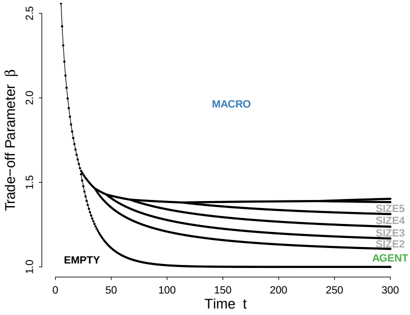


Predicting the Macroscopic Measurement

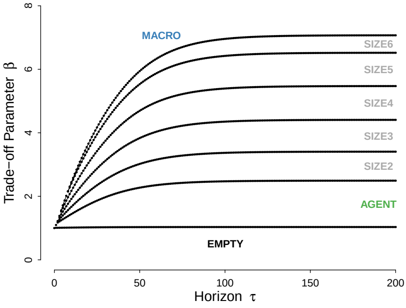


MACRO ($N = 7$)

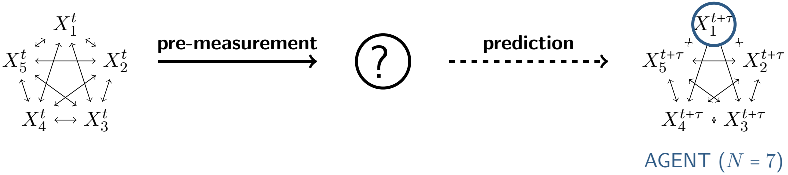
Fixed horizon $\tau = 3$



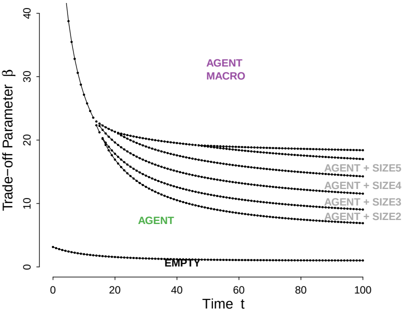
Fixed time $t = 100$



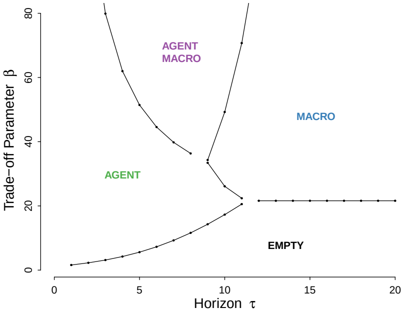
Predicting the Agent Measurement



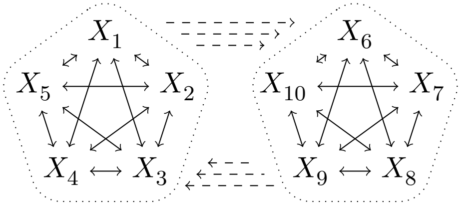
Fixed horizon $\tau = 3$



Fixed time $t = 0$



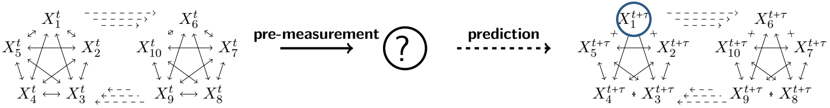
The Two-community Graph



First Community Ω_1 Second Community Ω_2

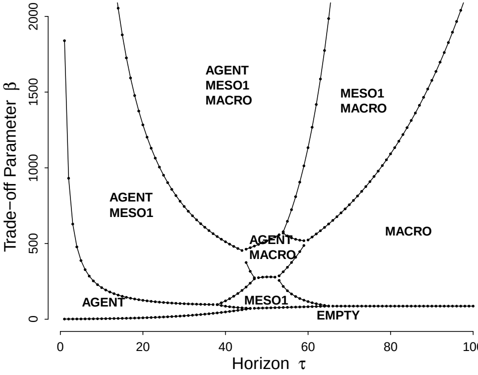
Coupling Parameter $\rho = \frac{\mathrm{Pr}(\text{inter edge})}{\mathrm{Pr}(\text{intra edge})} < 1$

Predicting the Agent Measurement



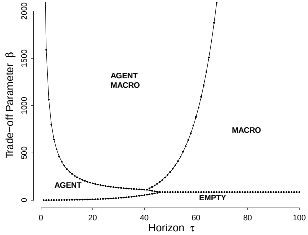
AGENT ($N = 10 + 10$ and $\rho = \frac{1}{5}$)

Fixed time $t = 0$

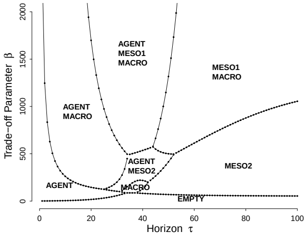


Some Other Heterogeneous Interaction Graphs

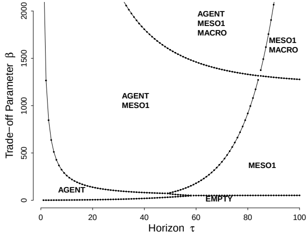
Homogeneous Case: $\rho_{1\to2} = \rho_{2\to1} = 1$



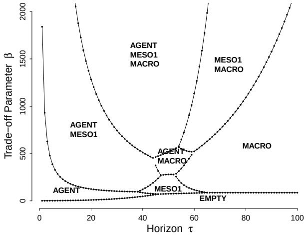
Follower Case: $\rho_{1\to2} = 1/5$ and $\rho_{2\to1} = 1$



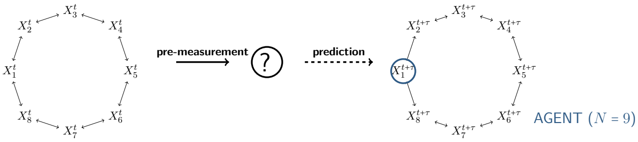
Leader Case: $\rho_{1\to2} = 1$ and $\rho_{2\to1} = 1/5$



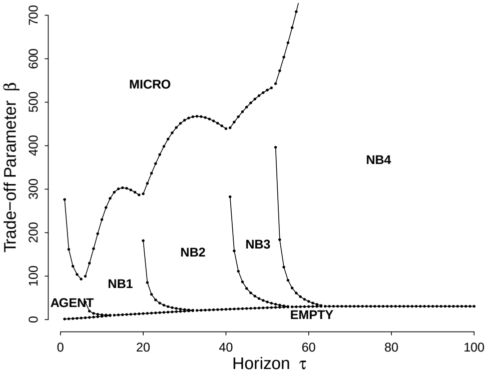
Symmetrical Case: $\rho_{1\to2} = \rho_{2\to1} = 1/5$



Predicting the Agent Measurement in the Ring



Fixed time $t = 0$



Application Perspectives

- Application Perspectives
  - Optimal prediction of **economic indicators** based on multilevel structures of the international trade network
  - Optimal prediction of **population dynamics** in ecology based on multilevel representation of the inter-species mutual dependencies
- Theoretical Perspectives
  - Taking into account the **problem of inferability** of low-level predictors when solving the optimisation problem (*e.g.*, Bayesian model selection)
  - Taking into account **domain-depend costs and rewards** as practical objective functions to be optimised in real-world scenarios

Thank you for your attention

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